RAMAKRISHNA MISSION VIDYAMANDIRA
(Residential Autonomous College under University of Calcutta)

B.A./B.Sc. FIFTH SEMESTER EXAMINATION, DECEMBER 2014
THIRD YEAR
MATHEMATICS (Honours)
Paper : V

Date : 20/12/2014
Time : 11 am – 3 pm
Full Marks : 100

[Use a separate Answer book for each Group]

Group – A

(*Answer **any five** questions*) [5×10]

1. a) Let G be a group of order pq where p and q are distinct primes. If G has a normal subgroup of order p and a normal subgroup of order q, prove that G is cyclic. [3]
   b) By making a composition table of $U(10)=\{\bar{1},\bar{3},\bar{7},\bar{9}\}$, prove that U(10) is a group under multiplication modulo 10. Is this group isomorphic to $\mathbb{Z}_4$? [4]
   c) Prove that the order of each element of the quotient group $\mathbb{Q}/\mathbb{Z}$ is finite. [3]
2. a) Prove that $\text{Aut}(\mathbb{Z}_2\times\mathbb{Z}_2)$ contains only six elements. [5]
   b) Let G be a finite commutative group. Let n be a positive integer such that gcd $(|G|,n)=1$. Show that $\phi:G\to G$ defined by $\phi(a)=a^n\ \forall a\in G$ is an isomorphism.
   Give an example to show that if gcd $(|G|,n)\neq 1$ then ϕ need not be an isomorphism. [5]
3. a) Let H and K be two normal subgroups of a group G and let G be an internal direct product of H and K. Prove that $G\cong H\times K$. [5]
   b) Prove that no group of order 56 is simple. [5]
4. a) Let G be a group of order n and p|n where p is prime. Show that G contains an element of order p. [5]
   b) If G is a group of order 96, prove that G has a normal subgroup of order 16 or a normal subgroup of order 32. [5]
5. a) State and prove Sylow's 2^{nd} theorem. [5]
   b) Let $T=\left\{\frac{a}{b}\in Q \mid \text{a and b are relatively prime and 5 does not divide b}\right\}$. Show that T is a ring under usual addition and multiplication. Also prove that $I=\left\{\frac{a}{b}\in T \mid 5 \text{ divides a}\right\}$ is an ideal of T. [5]
6. a) Prove that the rings $\mathbb{Z}[\sqrt{3}]$ and $\mathbb{Z}[\sqrt{5}]$ are not isomorphic. [3]
   b) Are the rings $\mathbb{Z}$ and $2\mathbb{Z}$ isomorphic? Justify your answer. [3]
   c) Let R be a ring with identity $1\neq 0$, such that R has no nontrivial left ideal. Show that R is a division ring. [4]
7. a) Prove that in an integral domain, every prime element is irreducible. [3]
   b) Prove that in a principal ideal ring, every irreducible element is prime. [3]
   c) In $\mathbb{Z}[i\sqrt{5}]$, show that the element 3 is irreducible but not prime. [4]
8. a) Let R be a commutarive ring with identity and M is an ideal of R. Prove that M is maximal iff R/M is a field. [5]
   b) Prove that the ideal <x> is prime in $\mathbb{Z}[x]$. [2]
   c) Show that $\mathbb{R}[x]/\langle x^2+1\rangle$ is a field. [3]

Group – B

(*Answer* ***any six*** *questions from* ***Q.No 9 - 17***) [6×5]

9. Show that the general error in interpolation is $R_{n+1}(x) = \omega(x)\dfrac{f^{n+1}(\xi)}{\lfloor n+1}$ where the symbols are of usual meaning. [5]

10. a) Show that $\Delta(\log f(x)) = \log\left\{1+\dfrac{\Delta f(x)}{f(x)}\right\}$ [3]

    b) $\Delta(\nabla) \equiv \delta^2$ [2]

    [Symbols have their usual meaning]

11. Establish numerical differentiation formula based on Lagrange's interpolation polynomial. [5]

12. Obtain Newton cotes formula for numerical integration. Deduce Simpson's one third rule from it. [3+2]

13. Describe the method of fixed point iteration for solving an equation f(x) = 0. Obtain the sufficient condition of convergence of the method. [4+1]

14. Describe the power method to calculate numerically the greatest eigen value of a real square matrix order n. [5]

15. Using Euler's modified method, obtain a solution of the differential equation :

$$\frac{dy}{dx} = x + \sqrt{y},\ y(0) = 1$$

for the range $0 \le x \le 0{\cdot}6$ in steps of 0·2. [5]

16. Solve the differential equation : $\dfrac{dy}{dx} = x^2 + y^2,\ y(0) = 1$

    by 4th order Runge-Kutta method from x = 0 to x = 0·2 with step length h = 0·1. [5]

17. Explain Gauss-Seidel method of solving a system of linear equations. Write down the sufficient condition of convergence of the method. [4+1]

(*Answer* ***any two*** *questions from* ***Q.No 18 – 20***) [2×10]

18. a) State and prove fundamental theorem of integral calculus. [4]

Or,

If $f:[0,1]\to\mathbb{R}$ is continuous on [0,1] then show that $\displaystyle\operatorname*{Lt}_{n\to\infty}\int_0^1 \frac{nf(x)}{1+n^2x^2}dx = \frac{\pi}{2}f(0)$.

    b) Prove or disprove :

    $f:[a,b]\to\mathbb{R}$ is a function of bounded variation on [a,b], then it is Riemann integrable. [3]

    c) Construct a real-valued function on a closed interval which is continuous, but not of bounded variation on that interval. [3]

19. a) If $\{f_n\}_n$ is a sequence of Riemann integrable functions and f is a function, all defined on [a,b], & $f_n \to f$ uniformly on [a,b], then prove that f must be Riemann integrable on [a,b], $\int_a^b f_n \to \int_a^b f$. [5]

b) Let $f, g : [0,1] \to \mathbb{R}$ be defined by

$$f(x) = \sin\left(\frac{\pi}{x}\right) \text{ if } 0 < x \le 1$$

$$= 0, \qquad x = 0$$

$$g(x) = 3x^2 + \cos x$$

Examine whether curve $\gamma = (f, g)$ is rectifiable. [3]

c) Show that $\left|\int_a^b \frac{\sin x}{x} dx\right| < \frac{4}{a}$, $0 < a < b$. [2]

20. a) Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Let the set of all discontinuity points of f in [a,b] have measure zero. Then prove that f must be Riemann integrable on [a,b] [4]

Or,

Let $f : [a,b] \to \mathbb{R}$ be a bounded function. If the set of points of discontinuity of f has a finite number of limit points in [a,b] then prove that f is Riemann integrable on [a,b].

b) Prove that : $\frac{\pi^3}{24\sqrt{2}} < \int_0^{\pi/2} \frac{x^2}{\sin x + \cos x} dx < \frac{\pi^3}{24}$ [3]

c) Evaluate the limit, $\underset{x \to 0}{\text{Lt}} \frac{\int_0^{x^2} e^{\sqrt{1+t}}\, dt}{x^2}$ [3]

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